A convergence proposal for the Collatz sequence: the geometric proof of the Custom Conjecture

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Abstract

In this work, the previous convergence proposal for the Collatz sequence is extended, introducing a geometric proof of the Custom Conjecture. The analysis focuses on the segments generated by the congruous numbers to 3 modulo 4 within the custom sequence, highlighting the angular behavior of the relative segments. It is shown that the average value of the angular coefficient of the successive segments of segment 1 is greater than the constant angular coefficient of segment 2 (equal to 0.5849625). This difference inevitably implies the intersection between the two lines, which takes place at a significant point: the convergence to the value 1 of the initial number N. This intersection provides a geometric confirmation of the validity of the custom Collatz conjecture.

1. Introduction

The Collatz sequence is defined for every positive integer$N$ as follows:

- If N is even: $N \to N/2$
- If N is odd: $N \to 3N + 1$

The conjecture states that, regardless of the initial value *N*, the sequence always converges to **1**. Despite its apparent simplicity, no general proof has yet been found.
The present paper is an extension of the paper entitled "A convergence proposal for the Collatz sequence: the Custom Conjecture", published in Academia.edu. In that paper, a reformulation of the famous Collatz conjecture was proposed, through a personalized sequence that preserves its structure but allows a more targeted analysis of the behavior of the natural numbers.
In this new version, the geometric aspect of the custom sequence is deepened, introducing a proof of the conjecture based on the intersection between two lines constructed from the segments associated with the terms of the sequence. In particular, the segments deriving from the congruous numbers 3 modulo 4 are analyzed, demonstrating that the average value of their angular coefficient exceeds the constant value of the second segment. Such a configuration inevitably leads to a point of intersection that represents convergence to the value 1, thus providing a geometric confirmation of the validity of Collatz's custom conjecture.

2. Custom Succession and Iterations

An alternative view of the sequence is proposed, based on **iterations composed of a certain number of**:

- Odd steps (application of $3N + 1$)
- Even steps (division by 2)

depending on the initial number of the iteration itself. This structure allows us to analyse the overall behaviour of the sequence in terms of growth and decline.

2. My personalised sequence and its simple open increasing broken sequence

Lemma (exceptional cases to be treated separately in my custom sequence)
The custom sequence defined below applies to all positive integers N , **with the exception** of the following three cases, which must be handled at the beginning of each iteration of the sequence (e.g. in any Python-type calculation programme, etc.):

1. **Powers of 2:** $N = 2^p$ with $p \geq 1$;
2. **Numbers of the form** $N = (2^p - 1)/3$ **:** with $p \geq 2$ and p even, i.e. the set $\{1; 5; 21; 85; 341; 1365; \dots\}$;
3. **Number** $N = 3$ **.**

The following properties apply in each case:

- **(1)** If $N = 2^p$, then the sequence converges to1 with **0 odd steps** and p **even steps** (successive divisions by 2) and in this initial iteration the number of odd steps is $a_1 = 0$ and the number of even steps is $b_1 = p$, so the difference $b_1 - a_1$ has increased by p.
- **(2)** If $N = \frac{2^p-1}{3}$ with $p \geq 2\ e\ p$ equal to , then applying a single odd step gives $3N + 1 = 2^p$. From here, the sequence converges to1 after **1 odd step** and$p$ **even steps**, and in this iteration**,** the number of odd steps is $a_i = 1$ and the number of even steps is $b_i = p$, so the difference $b_i - a_i$ has increased by $p - 1$.
- **(3)** If $N = 3$, the canonical sequence is
$$3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1,$$
so convergence to 1 occurs with **2 odd steps** and **5 even steps** in this iteration ***i*** the number of odd steps is $a_i = 2$ and the number of even steps is $b_i = 5$, so the difference $b_i - a_i$ has increased by 3 .

Key observations

Since $N = 2^p$ is even, it must be the initial number of the first iteration in the custom sequence, while all the **initial numbers of the iterations following the first** are always **odd numbers,** necessarily **congruent to 1 or 3 modulo 4.**
Furthermore, the numbers $N = \frac{2^p-1}{3}$ with $p \geq 2\ e\ p$ equal (there are not many) can be initial numbers of any iteration, both initial and intermediate, in which the sequence immediately converges to 1. In fact, the number $N =$ 5 = $\frac{2^4-1}{3}$ with p=4 is a number that always results in in the last iteration (most of the time it does not appear in the last iteration of the sequence because it follows the number $N = 3$ which 'overshadows' it which converges to 1 with 1 odd step and 4 even steps.
On the other hand, the number $N =$ 3 is a number that always appears in the last iteration of the sequence that converges to 1 with 2 odd steps and 5 even steps, incorporating the number $N =$ 5, which is the second number of the congruent type 1 modulo 4 (the first is 1), while the number $N =$ 3 is the first number of the congruent type 3 modulo 4.
In the custom sequence, after a possible even initial number, **all subsequent iterations have only odd numbers congruent to 1 or 3 modulo 4 as their initial number**, except for the explicit exceptions indicated above in the lemma.

Let us examine with the following theorems how my custom sequence develops in each iteration, considering the 4 types of numbers N congruent mod. 4, excluding the three exceptional cases seen previously in the lemma.

Theorem 3.1 ($N \equiv 0(mod4)$**)**

Let N be a positive integer such that $N \equiv 0\ (mod4)$. Then we can write $N = 2^p(2k+1)$ with $p \geq 2$ and $k \geq 0$ integers; by performing the first p steps of the Collatz sequence (i.e., dividing by 2 each time the term is even), we obtain the term $N[p] = 2k+1$, which is odd and strictly less than N . Furthermore, $2k+1 \equiv 1(mod4)$ if k is even and $2k+1 \equiv 3(mod4)$ if k is odd; at this initial stage, the number of odd steps is $a_1 = 0$ and the number of even steps is $b_1 = p$, so the difference $b_1 - a_1$ has increased by p .
Proof:
Since $N \equiv 0(mod4)$, N is even and at least divisible by 4 . There is therefore a decomposition $N = 2^p m$ with $p \geq 2$ and m odd. Writing $m = 2k+1$ with $k \geq 0$, we obtain

$$N = 2^p(2k+1).$$

In the Collatz sequence, if the term is even, the operation $n \mapsto n/2$ is applied. Applying this operation p times in succession to the number $N = 2^p(2k+1)$ eliminates the factor 2^p and we obtain

$$N^{[p]} = \frac{N}{2^p} = 2k+1.$$

Because $p \geq 2$ we have $2^p \geq 4$, therefore

$$N^{[p]} = 2k+1 = \frac{N}{2^p} < \frac{N}{4} < N,$$

i.e. the term obtained after the p steps is odd and is strictly less than the initial value N .

2. As for the congruence modulo 4 of the new initial term of the next iteration: if k is even, $k = 2h$ and $2k+1 = 4h+1 \equiv 1(mod4)$; if k is odd, $k = 2h+1$ and $2k+1 = 4h+3 \equiv 3(mod4)$. Therefore, $2k+1 \equiv 1(mod4)$ when k is even and $2k+1 \equiv 3(mod4)$ when k is odd. Finally, in these first p steps, the odd operator $3n+1$ was never applied: only p even steps (divisions by 2) were performed. Therefore, the count of steps in the initial iteration alone is $a_1 = 0$ (odd steps) and $b_1 = p$ (even steps), and the initial difference $b_1 - a_1$ is p compared to the initial situation where, counting only this single move, the difference was 0 . Q .E.D.

Theorem 3.2 ($N \equiv 2(mod4)$ **)**

Let N be an even natural number such that $N \equiv 2(mod4)$. Then N can be written in the form

$$N = 4k+2 \qquad (k \geq 0).$$

The next term in the Collatz sequence after an even step (i.e., applying the rule $n \mapsto \frac{n}{2}$ only once) is

$$N_1 = \frac{N}{2} = 2k+1,$$

which is an odd number strictly less than N . Furthermore:

1. If k is even, then $2k+1 \equiv 1(mod4)$.
2. If k is odd, then $2k+1 \equiv 3(mod4)$.
3. In this iteration (from N to N_1), the odd steps are $a_1 = 0$ and the even steps are $b_1 = 1$; therefore, the difference $b_1 - a_1$ has increased by 1 .

Proof

1. Since $N \equiv 2(mod4)$, by definition there exists$k \in N$ with $N = 4k+2$. Dividing by 2 we obtain

$$N_1 = \frac{N}{2} = 2k+1.$$

Since $2k+1$ is odd. Comparing N_1 with N :

$$N - N_1 = (4k + 2) - (2k + 1) = 2k + 1 > 0,$$

therefore $N_1 < N$.

3. We evaluate the congruences modulo 4:
   - If k is even, $k = 2m$ for some m ; then
     $$2k + 1 = 2(2m) + 1 = 4m + 1 \equiv 1(mod4).$$
   - If k is odd, $k = 2m + 1$; then
     $$2k + 1 = 2(2m + 1) + 1 = 4m + 3 \equiv 3(mod4).$$
4. We count the even and odd steps in the single iteration $N \to N_1$. We start with an even number N and apply the even rule $n \mapsto n/2$ only once; therefore, in this iteration there is one even step ($b_1 = 1$) and no odd steps ($a_1 = 0$). Therefore, the difference $b_1 - a_1 = 1$, i.e. it has increased by one unit compared to the initial situation in which, counting only this single move, the difference was 0 . Q.E.D.

Theorem 3.3 (third type, $N \equiv 1(mod4)$; change of orbit and conservation of the balance of steps)

Let $N = 4h_0 + 1$ be a positive integer and let $N' = h_0$. Then:

1. After an odd step followed by two even steps applied to N , we obtain
   $$N[3] = 3h_0 + 1.$$
2. If h_0 is odd, then$N$ and N' are isopath: N reaches $N[3] = 3h_0 + 1$ in three steps (D, P, P) while N' reaches $N[3] = 3h_0 + 1$ in a single step (D). Their orbits converge at $N[3]$ and, therefore, any convergence property valid for one of the three numbers N, N'', N [3] also applies to the other two. I call this situation an "orbit change".
3. In the case where h_0 is odd, replacing $N[3] = 3h_0 + 1$ with $N' = h_0$ corresponds to the formal elimination of an odd step in the temporal description of the trajectory, but does not alter the overall balance of even and odd steps with respect to the canonical Collatz sequence.
4. If h_0 is even, there is no change of orbit and the sequence continues canonically from N to$N[3] = 3h_0 + 1$ in three steps (D, P, P) without a change of orbit.

Proof

Let $N = 4h_0 + 1$.

1. Direct calculation of $N[3]$. Applying the odd rule to N , we have

$$3N + 1 = 3(4h_0 + 1) + 1 = 12h_0 + 4 = 4(3h_0 + 1).$$

Since this number is a multiple of 4 , applying two divisions by 2 (two even steps) we obtain

$$\frac{4(3h_0 + 1)}{2^2} = 3h_0 + 1.$$

Therefore, the term obtained after the three steps (D, P, P) is

$$N[3] = 3h_0 + 1.$$

2. Intersection with $N' = h_0$ when h_0 is odd

If h_0 is odd, applying the odd rule to N' , we immediately obtain

$$N'[1] = 3h_0 + 1 = N[3].$$

Therefore, N reaches $N[3]$ in three steps, while N' reaches it in just one: the orbits converge at $N[3]$ and therefore $N,\ N'\ e\ N[3]$ are isopathic.

3. Explicit 'elimination' of the odd step

To make the situation more explicit, we observe that h_0 is directly the odd preimage of $3h_0 + 1$ with respect to the inverse odd operation of Collatz:

$$h_0 = \frac{(3h_0 + 1) - 1}{3}.$$

This equality shows that, in the transition from N to $N[3]$, the odd contribution that leads to $3h_0+1$ is the same odd step that, if starting from N', is performed as the first step. Consequently, replacing $N[3]=3h_0+1$ with $N'=h_0$ in the description of the trajectory formally eliminates the initial odd step in the history of N (so that only the two odd steps are counted) because that step will be 'incorporated' as the first odd step of the next iteration starting from N'. Conservation of step balance (strict interpretation) Let us compare the counts up to the intersection point $N[3]$:

- Path from N to N': sequence of step types P, P, therefore
$$\#odd=0, \#even\ =2.$$
- Path from N' to $N[3]$: sequence D, then
$$\#dispari=1, \#pari\ =0$$
- Path from N to $N[3]$: sequence D, P, P, then
$$\#odd=1, \#even\ =2$$

The formula $h_0=\frac{(3h_0+1)-1}{3}$ shows that the odd step present in both descriptions is the same arithmetic event: it does not disappear, only the iteration index in which it is counted changes. If we reconstruct the unique trajectory starting from the intersection point and extend it downstream, the total number of odd and even steps taken by the pair of orbits up to any subsequent common term remains unchanged and identical to the canonical Collatz sequence. In other words, the apparent 'removal' of the odd step in the orbit change is purely a difference in temporal parameterisation: the contribution to the overall balance of steps is preserved. So in this iteration, there are two even steps ($b_i=2$) and no odd steps ($a_i=0$). Therefore, the difference $b_i-a_i=2$ has increased by two units compared to the situation in the previous iteration.

4. Case h_0 even

If h_0 is even, there is no change of orbit and the sequence continues in a canonical manner from N to $N[3]=3h_0+1$ without a change of orbit in three steps (D, P, P) and where the initial number of the next iteration is odd and therefore congruent to 1 or 3 (mod4). Therefore, in this iteration, there are two even steps ($b_i=2$) and one odd step ($a_i=1$). Thus, the difference $b_i-a_i=1$ has increased by one unit compared to the previous iteration. Q.E.D.

Final observation

This explanation clearly shows that in the case $N\equiv 1(mod4)$ where $N=4h_0+1$ with h_0 odd, the replacement of $3h_0+1$ in the iteration with $h_0=\frac{(3h_0+1)-1}{3}$ is the key to the apparent 'elimination' of the odd step: it is not lost, but simply considered as the first odd step of the trajectory starting fromN' in the next iteration. For the study of the convergence of $N=4h_0+1$, it is therefore legitimate to replace without altering the overall count of odd and even steps.

Theorem 3.4 (fourth type,$N\equiv 3(mod4)$)

Let N = 4 h_0+3 which can be written as $N=2^p(1+2k)-1$ with $p\geq 2$ and $k\geq 0$ (therefore $N\equiv 3(mod4)$) as it precedes the number $N+1\equiv 0(mod4)$ **).** Then, applying the Collatz sequence:

1. After $2(p-1)$ double steps, we obtain
$$N[2(p-1)]\ =\ 3^{p-1}\,2\,(1+2k)-1;$$
2. this term is always congruent to $1(mod4)$ and can be written as $N[2(p-1)]=4h_1+1$ with
$$h_1=\frac{3^{p-1}(1+2k)-1}{2}\in N;$$
3. Finally, if h_1 is even, then, by **Theorem 3.3**, after 3 steps, one of which is odd and two of which are even, we have
$$N[2p+1]=3h_1+1,$$
while if h_1 is odd, then, by **Theorem 3.3 ,** after 2 steps, both of which are even, we have

$$N[2p] = h_1.$$

Proof

(Part 1 — double steps)

Let's start with $N = 2^p(1+2k) - 1$. Since N is odd, we apply the so-called *double step* (apply the odd rule $3n+1$ followed by division by 2 when possible). We calculate the first double step:

$$N[2] = \frac{3(2^p(1+2k)-1)+1}{2} = 3 \cdot 2^{p-1}(1+2k) - 1.$$

The result is still odd (the form is $3 \cdot 2^{p-1}(1+2k) - 1$), so we can repeat the same reasoning: at each double step, the exponent of 3 increases by 1 and the exponent of 2 decreases by 1 , while the factors $(1+2k)$ and the term -1 remain unchanged. Iterating until the exponent of 2 becomes 1 (i.e. after$p-1$ double steps), we obtain:

$$N[2(p-1)] = 3^{p-1}\,2\,(1+2k) - 1,$$

which proves part (1).

(Part 2 — congruence modulo 4 and form $4h_1 + 1$)

Now let us consider $M := N[2(p-1)] = 3^{p-1}\,2\,(1+2k) - 1$. We want to establish the class modulo 4 of M .

Suppose, by contradiction, that $M \equiv 3(mod 4)$, i.e. that there exists h such that $M = 4h + 3$. Then

$$h = \frac{M-3}{4} = \frac{3^{p-1}\,2\,(1+2k) - 4}{4} = \frac{3^{p-1}(1+2k)}{2} - 1.$$

But $3^{p-1}(1+2k)$ is the product of two odd numbers and is therefore odd; therefore, the fraction $\frac{3^{p-1}(1+2k)}{2}$ is not an integer, contradicting the fact that$h$ must be an integer. Hence, $M\ non\ è\ \equiv 3(mod 4)$. Since M is odd, the only remaining possibility modulo 4 is $M \equiv 1(mod 4)$. Therefore, there exists an integer h_1 with

$$M = 4h_1 + 1.$$

Let us explicitly calculate h_1 :

$$h_1 = \frac{M-1}{4} = \frac{3^{p-1}\,2\,(1+2k) - 2}{4} = \frac{3^{p-1}(1+2k) - 1}{2},$$

which is an integer because $3^{p-1}(1+2k)$ is odd, so subtracting 1 becomes even and divisible by 2 . This proves part (2).

(Part 3 — what happens next: dependence on the parity of h_1)

Now let's apply Collatz's rule to the term $M = 4h_1 + 1$. If M is written as $4h_1 + 1$, then the next step (odd step) calculates $3M + 1 = 12h_1 + 4 = 4(3h_1 + 1)$; dividing by 2 appropriately according to the sequence of steps considered, we have $M[3] = (3h_1 + 1)$, and consequently we obtain the two classic cases indicated in **<u>Theorem 3.3</u>**:

- If h_1 is **even**, then $h_1 = 2t$ and $M = 4(2t) + 1$. Applying the odd rule, we obtain $3M + 1 = 12(2t) + 4 = 4(6t + 1)$: following the sequence of divisions by 2 required by the steps, the useful result obtained after the indicated steps is

$$N[2p+1] = 3h_1 + 1.$$

(This corresponds to the pattern given in the original formulation: if h_1 is even ->$N[2p+1] = 3h_1 + 1$.)

- If h_1 is **odd**, then there is an 'orbit change', i.e. the number $N[2p] = h_1$ is replaced in the customised sequence at $N[2p+1] = 3h_1 + 1$ with the reduction of the first odd step, so that with only 2 even steps we have

$$N[2p] = h_1.$$

(This corresponds to the original scheme: if h_1 is odd ->$N[2p] = h_1$)

Both conclusions follow from the direct application of the rule $3n+1$ followed by divisions by 2 as long as possible, noting that the representation $M = 4h_1 + 1$ makes clear the form of the subsequent results as a function of the parity of h_1 . Q.E.D.

<u>Theorem 3.5</u> (extension of <u>Theorem 3.4</u>)

This theorem applies in the customised sequence to each initial number of the iteration $\equiv 3\ (mod4)$ **and makes it faster.**
Let
$$N = 2^p(1+2k)-1, p \geq 2, k \geq 0,$$
and, as in the previous theorem **Theorem 3.4**,
$$M := N[2(p-1)] = 3^{p-1}\, 2(1+2k) - 1.$$
We define
$$h_1 := \frac{3^{p-1}(1+2k)-1}{2} \in Z.$$
Then
1. $M \equiv 1(mod4)$a nd indeed $M = 4h_1 + 1$.
2. The parity of h_1 is determined by the parity of p and k
   - h_1 is **even** if and only if p and k have opposite parity;
   - h_1 it is **odd** if and only if p and k have the same parity.
3. If h_1 is **even**, we obtain $N[\,2p+1\,] = 3h_1 + 1$ and therefore $N[\,2p+1\,] = \frac{3^p(1+2k)-1}{2}$, , so in this iteration we have *p+1* even steps ($b_i = p+1$) and p odd steps ($a_i = p$). Therefore, the difference $b_i - a_i = 1$ has increased by one unit compared to the previous iteration.
4. If h_1 is **odd**, we obtain $N[2p] = h_1 = \frac{3^{p-1}(1+2k)-1}{2}$ ("change of orbit"), so in this iteration we have *p+1* even steps ($b_i = p+1$) and *p-1* odd steps ($a_i = p-1$). Therefore, the difference $b_i - a_i = 2$ has increased by two units compared to the previous iteration.

Proof: (1) and definition of h_1 .
From the previous theorem, we know
$$M = 3^{p-1}\, 2(1+2k) - 1.$$
Writing $M = 4h_1 + 1$, we obtain, by definition,
$$h_1 = \frac{M-1}{4} = \frac{3^{p-1}2(1+2k)-2}{4} = \frac{3^{p-1}(1+2k)-1}{2},$$
which is an integer because $3^{p-1}(1+2k)$ is odd; therefore, subtracting 1 we obtain an even number.
(2) parity of h_1 .
To determine the parity of h_1 , we consider $A := 3^{p-1}(1+2k)\ (mod4)$. Since $3 \equiv -1(mod4)$, we have
$$3^{p-1} \equiv (-1)^{p-1}(mod4),$$
i.e. $3^{p-1} \equiv 1$ if$p$ is odd, otherwise $\equiv 3$ if$p$ is even. Furthermore, $1+2k \equiv 1$ if$k$ is even, otherwise $\equiv 3$ if k is odd. Therefore, the product A is equal to 1 modulo 4 exactly when the two factors have the same class modulo 4 (i.e. when one is 1 and the other is 1 , or both are 3), and is equal to 3 modulo 4 in the other cases.
Now
$$h_1 = \frac{A-1}{2}(mod2)$$
and therefore:
- If $A \equiv 1(mod4)$ then $A - 1 \equiv 0(mod4)$ and h_1 is even;
- If $A \equiv 3(mod4)$ then $A - 1 \equiv 2(mod4)$ and h_1 is odd.

From the various possible combinations, we obtain that h_1 **is even if p and k have opposite parity**, while h_1 **is odd if p and k have the same parity**. This proves (2).
(3) Case h_1 is even.
If h_1 is even, then applying the odd rule to $M = 4h_1 + 1$:
$$3M + 1 = 12h_1 + 4 = 4(3h_1 + 1).$$
This quantity is divisible by 4 ; performing two successive divisions by 2, we obtain
$$\frac{4(3h_1+1)}{4} = 3h_1 + 1.$$
Calculating $3h_1 + 1$ with the expression of h_1 , we obtain

$$N[2p+1] = 3h_1+1 = 3\cdot\frac{3^{p-1}(1+2k)-1}{2}+1 = \frac{3^p(1+2k)-1}{2},$$

i.e. the statement in point (3) whereby if at the beginning of iteration ***i*** we have a number

$$N = 2^p(1+2k)-1, p \geq 2, k \geq 0,$$

with ***p* and *k* having different parity,** the initial number of the next iteration ***i+1*** is

$$N[2p+1] = \frac{3^p(1+2k)-1}{2}$$

Count of steps: up to M, there were $p-1$ odd steps and $p-1$ even steps (i.e. $2(p-1)$ "double steps"); from the transformation of M to $3h_1+1$, 1 odd step (the application $3n+1$) and 2 even steps (the two divisions by 2) are added, so the total becomes p odd steps and $p+1$ even steps. Therefore, the difference $b_i - a_i = 1$ has increased in this iteration ***i*** by one unit compared to the situation in the previous iteration.

(4) case h_1 odd

If h_1 is odd according to **<u>Theorem 3.3</u>**, there is an 'orbit change', so the number $N[2p+1] = 3h_1+1$ is replaced by the number $N[2p] = h_1 = \frac{3^{p-1}(1+2k)-1}{2}$ reduced by one odd step, and this number becomes the initial number of the next iteration, i.e. the statement in point (4) has been verified.

Therefore, if at the beginning of iteration ***i*** we have a number

$$N = 2^p(1+2k)-1, p \geq 2, k \geq 0,$$

with ***p* and *k* having the same parity,** the initial number of the next iteration ***i+1*** is

$$N[2p] = \frac{3^{p-1}(1+2k)-1}{2}.$$

Count of steps: up to M, there were $p-1$ odd steps and $p-1$ even steps ($2(p-1)$ r "double steps"); from the "orbit change" from $3h_1+1$ toh_1, only the 2 even steps are added, so the total becomes $p-1$ odd steps and $p+1$ even steps. Therefore, the difference $b_i - a_i = 2$ has increased in this iteration ***i*** by two units compared to the situation in the previous iteration. Q.E.D.

From the previous theorems, it follows that the following statements are true in this customised sequence:

A. the exceptional cases verify the conjecture;
B. the transformation of numbers in each iteration faithfully follows only one of the cases described above in the previous theorems based on the initial number of each iteration;
C. from the first iteration onwards, the total number of even steps (which I indicate with b) is always greater than the total number of odd steps (which I indicate with a), i.e. b > a, and in subsequent iterations, b increases more than a;
D. the difference between the total number of even steps and the total number of odd steps always increases by 1 or 2 units in each central iteration that has as its initial number a number congruent with 1 or 3 modulo 4, and this produces in subsequent iterations a difference between b and a that always increases;
E. the value of the initial numbers congruent with 1 modulo 4 in each iteration decreases, while those congruent with 3 modulo 4 increase their value in most cases.

Construction of the open and simple broken line obtained from the results of the customised sequence

Preliminary definitions

Let *N* be a positive integer and consider the sequence (a_i, b_i) generated by the custom sequence consisting only of odd numbers congruent to 1 or 3 modulo 4:

- a_i is the total number of odd steps taken up to and including iteration *i*;
- b_i is the total number of even steps taken up to and including iteration *i*;

• The difference $b_i - a_i$ is their difference at iteration i, which always increases by 1 or 2 units at each iteration.

We define the broken line in the Cartesian plane formed by the segments connecting the points

$$P_{i-1} = (a_{i-1}, b_{i-1} - a_{i-1})\ con\ P_i = (a_i, b_i - a_i), \forall i \geq 1$$

obtained in the customised sequence with starting point

$$P = (0,0).$$

3. When N *converges to 1*

Definition.

Let $N \in N$ be the initial number of the custom sequence.

Each iteration $i \geq 1$ is associated with two integers (a_i, b_i) such that:

$$b_i > a_i \forall i$$

and the convergence of *N* to 1 occurs when, for a certain index s , the following is true:

$$\frac{3^{a_s} N}{2^{b_s}} < 1.$$

Transition to logarithms.

The inequality is equivalent to:

$$a_s log\ (3) - b_s log\ (2) + log\ (N) < 0,$$

from which we obtain the two conditions:

$$b_s > a_s \frac{log\ (3)}{log\ (2)} + log\ _2(N), b_s - a_s > a_s \frac{log\ (3/2)}{log\ (2)} + log\ _2(N).$$

Approximate values.

Since

$$\frac{log\ (3)}{log\ (2)} \approx 1.5849625, \frac{log\ (3/2)}{log\ (2)} \approx 0.5849625,$$

it follows that:

$$b_s > a_s \cdot 1.5849625 + log\ _2(N), b_s - a_s > a_s \cdot 0.5849625 + log\ _2(N).$$

Rounding.

Since $b_s,\ b_s - a_s \in \mathbb{N}$, we define rounding to the nearest integer as:

$$b_s = \lceil a_s \cdot 1.5849625 + log\ _2(N) \rceil, b_s - a_s = \lceil a_s \cdot 0.5849625 + log\ _2(N) \rceil.$$

Discrete function associated with the convergence of N to 1 .

We define the discrete function:

$$f(a_i) = \lceil a_i \cdot 0.5849625 + log\ _2(N) \rceil,$$

which represents the "discrete growth" of the values $b_i - a_i$ and is the condition that verifies the convergence of *N* to 1 for a particular value of a_i .

Geometric representation.

In the Cartesian plane, we consider the straight line:

$$y = a_i \cdot 0.5849625 + log\ _2(N),$$

with slope

$$m = 0.5849625,$$

and intercept

$$Q = (0, log\ _2(N)).$$

The graph of the discrete function f is an **open, simple, increasing broken** line , which has a similar trend to the increasing straight line y formed by segments connecting the points:

$$Q_{i-1} = \left(a_{i-1}, f(a_{i-1})\right)\ con\ Q_i = (a_i, f(a_i)), \forall i \geq 1.$$

5. Study of the open and simple line obtained from the results of the personalized sequence

We know that the open and simple line obtained from the results of the custom sequence (which we will call **line 1**) is formed by increasing segments, each with **an inclination *m*** dependent on one of the six points of the custom sequence. We then compare the variable **slope m** of line 1 with the constant slope of line 2 of value **0.5849625 , as follows:**

- **Point 1: N≡ 0 mod 4** where **N=2^p·(1+2k) is even**
  - **N must be the initial number of the first iteration**
  - After **p even passes**, the next of N is **1+2k odd**
  - S → 1+2K≡ 1 Mode 4
  - If odd k → 1+2k≡ 3 mod 4
  - Odd steps: a=0 , even: b=p, difference b-a=p
  - The vertical segment of line 1 joins the point (0,0) with the point (0,p) and is increasing with **inclination *m* = p/0= +∞**, so ***m* > *0.5849625*.**
  - The end point of the iteration (0,p) becomes the starting point of the next segment and it is **closer** to the starting point $(0, \log_2(N))$ of the line 2 which remains unchanged since a=0.

- **Point 2: N≡ 2 mod 4** where **N=4*k+2 is even**:
  - **N must be the initial number of the first iteration**
  - After **1 even pass**, the next of N is **2k+1 odd**
  - S Pari → 2K +1≡ 1 Mode 4
  - If odd k → 2k +1≡ 3 mod 4
  - Odd passes: a=0, even: b=1, difference b-a=1
  - The vertical segment of line 1 joins the point (0,0) with the point (0,1) and is increasing with **inclination *m* = 1/0= +∞**, so ***m* > *0.5849625*.**
  - The end point of the iteration (0,1) becomes the starting point of the next segment and it is **closer** to the starting point $(0, \log_2(N))$ of the line 2 which remains unchanged since a=0.

- **Point 3: N≡ 1 mod 4,** where **N=4·(1+2k)+1 is odd with 1+2*k odd:**
  - **N can be the initial number of each iteration i including the first**
  - After **0 odd passes + 2 even passes**, the next of N is **1+2k odd**
  - S → 1+2K≡ 1 Mode 4
  - If odd k → 1+2k≡ 3 mod 4
  - Odd passes: a= 0, even: b=2, difference b-a=2
  - The vertical segment of line 1 joins the point with the point and is increasing with $(a_i, b_i - a_i)$ $(a_i, b_i - a_i + 2)$**an inclination *m* = 2/0= +∞**, so ***m* > *0.5849625*.**
  - The end point of iteration $(a_i, b_i - a_i + 2)$**i** becomes the start point of the next segment and it is **closer** to the point $(a_i, f(a_i))$ of line 2.

- **Point 4: N≡ 1 mod 4, where N=4·(2k)+1 is odd with 2*k even:**
  - **N can be the initial number of each iteration i including the first**
  - After **1 odd pass + 2 even passes**, the next of N is **6k+1**
  - From: → 6K+1≡ 1 Mode 4
  - If k odd → 6k+1≡ 3 mod 4
  - Odd steps: a = 1, even: b = 2, difference b-a=1
  - The oblique segment of the line 1 joins the point with the point and is increasing with $(a_i, b_i - a_i)$ $(a_i + 1, b_i - a_i + 1)$**an inclination *m* = 1/1 = 1**, so ***m* > *0.5849625*.**

- The end point of iteration $(a_i + 1, b_i - a_i + 1)$**i** becomes the start point of the next segment and it is **closer** to the point $(a_i + 1, f(a_i + 1))$ of line 2.

◆ **Point 5: N≡ 3 mod 4 where N=2^p· (1+2k)−1 with k and p having the same parity and with k≥0** and **p≥2**

- **N can be the initial number of each iteration i including the first**
- After **p−1 odd passes** and **p+1 even passes**, the next of N is **[3^(p−1) · (1+2k)−1]/2**
- Odd passes: a=p−1, even: b=p+1, difference b-a=2
- The oblique segment of the line 1 joins the point with the point and is increasing with $(a_i, b_i - a_i)$ $(a_i + p - 1, b_i - a_i + 2)$**a variable inclination *m* = 2/(p-1)**. We have ***m* > *0.5849625* if 2<=p<=4 or *m* < *0.5849625* if p>=5.**
- The end point of iteration $(a_i + p - 1, b_i - a_i + 2)$**i** becomes the starting point of the next segment and it is **closer** to $(a_i + p - 1, f(a_i + p - 1))$ the point of the line 2 **if 2<=p<=4** while it is **not very close** to the point $(a_i + p - 1, f(a_i + p - 1))$**if p>=5.**

◆ **Point 6: N≡ 3 mod 4 where N=2^p· (1+2k)−1 with k and p having different parity and with k≥0** and **p≥2**

- **N can be the initial number of each iteration i including the first**
- After **p odd passes** and **p+1 even passes** , the next of N is **[3^p · (1+2k)−1]/2**
- Odd passes: a=p, even: b=p+1, difference b-a=1
- The oblique segment of the line 1 joins the point with the point and is increasing with $(a_i, b_i - a_i)$ $(a_i + p, b_i - a_i + 1)$**a variable inclination *m* = 1/p**. We have ***m* < *0.5849625* for each value of p.**
- The end point of iteration $(a_i + p - 1, b_i - a_i + 2)$**i** becomes the start point of the next segment and it is **not very close** to the point $(a_i + p, f(a_i + p))$ of line 2 **for any value of p**.

Thus it can be seen that in the custom sequence only **the points 5 and 6**, which concern the numbers **N≡ 3 mod 4, have the segments of segment 1 with an inclination that is always positive but not always greater than the corresponding segments of segment 2.**

6. Numbers N ≡ 3 mod.4

The number N ≡ 3 mod.4 is odd and is written $\boldsymbol{N = 2^p * (1 + 2 * k) - 1}$ with p>=2 and k>=0 integers. For example, if **p=20** we have $\boldsymbol{N = 2^{20} * (1 + 2 * k) - 1 = 2097152 * k + 1.048.575}$ with k>=0. It is important to know how many N there are that lead to a certain multiplicity p for the 2 of N+1 since it is the N ≡ 3 mod.4 that, based on the value of p and its parity with k, delay the intersection between the two lines. The numbers N ≡ 3 mod.4 which have multiplicity **p** are written $\boldsymbol{N = 2^p - 1 + 2^{(p+1)} * k}$ for each **p≥2** and **k≥0** integers.
From **Tables 1** and **2** below, it is shown which and how many are for each value of p the respective **m** in a group of numbers to be **0** a $\boldsymbol{2^{20} - 1 = 1048575}$. It should be remembered that the numbers of type N ≡ 3 mod.4 are 1/4 of all the 1048576 considered (i.e. 262144) and that of them half 131072 belong to point 5 (Table 1) and the other half 131072 belong to point 6 (Table 2) of the custom sequence.

Table 1

Point 5 of the sequence with p and k the same parity				
p=	**N=**	**m=2/(p-1)**	**Weight of m with p and k same parity**	**Weight in percentage on 1048576 numbers**
2	**N=3+8*k**	**2.0000**	**65536**	**6.250000%**
3	**N=7+16*k**	**1.0000**	**32768**	**3.125000%**
4	**N=15+32*k**	**0.6667**	**16384**	**1.562500%**
5	**N=31+64*k**	**0.5000**	**8192**	**0.781250%**

6	N=63+128*k	0.4000	4096	0.390625%
7	N=127+256*k	0.3333	2048	0.195313%
8	N=255+512*k	0.2857	1024	0.097656%
9	N=511+1024*k	0.2500	512	0.048828%
10	N=1023+2048*k	0.2222	256	0.024414%
11	N=2047+4096*k	0.2000	128	0.012207%
12	N=4095+8192*k	0.1818	64	0.006104%
13	N=8191+16384*k	0.1667	32	0.003052%
14	N=16383+32768*k	0.1538	16	0.001526%
15	N=32767+65536*k	0.1429	8	0.000763%
16	N=65535+131072*k	0.1333	4	0.000381%
17	N=131071+262144*k	0.1250	2	0.000191%
18	N=262143+524288*k	0.1176	1	0.000095%
19	N=524287+1048576*k	0.1111	0	0.000000%
20	N=1048575+2097152*k	0.1053	1	0.000095%
		Total	131072	12.50%

Table 2

Point 6 of the sequence with p and k different parity				
p=	N=	m=1/p	Weight of m with p and k different parity	Weight in percentage on 1048576 numbers
2	N=3+8*k	0.5000	65536	6.250000%
3	N=7+16*k	0.3333	32768	3.125000%
4	N=15+32*k	0.2500	16384	1.562500%
5	N=31+64*k	0.2000	8192	0.781250%
6	N=63+128*k	0.1667	4096	0.390625%
7	N=127+256*k	0.1429	2048	0.195313%
8	N=255+512*k	0.1250	1024	0.097656%
9	N=511+1024*k	0.1111	512	0.048828%
10	N=1023+2048*k	0.1000	256	0.024414%
11	N=2047+4096*k	0.0909	128	0.012207%
12	N=4095+8192*k	0.0833	64	0.006104%
13	N=8191+16384*k	0.0769	32	0.003052%
14	N=16383+32768*k	0.0714	16	0.001526%
15	N=32767+65536*k	0.0667	8	0.000763%
16	N=65535+131072*k	0.0625	4	0.000381%
17	N=131071+262144*k	0.0588	2	0.000191%
18	N=262143+524288*k	0.0556	1	0.000095%
19	N=524287+1048576*k	0.0526	1	0.000095%
20	N=1048575+2097152*k	0.0500	0	0.000000%
		Total	131072	12.50%

For both tables 1 and 2 the weighted mean value, the variance and the standard deviation are calculated, the following results are obtained:

For **table 1** we have:

- ***Media ponderata*: 1.386298**
- ***Varianza ponderata*: 0.407146**
- ***Standard deviation* (standard deviation): 0.638080**

For **table 2** we have:

- ***Media ponderata*: 0.386288**
- ***Varianza ponderata*: 0,015258**

- ***Standard deviation* : 0.123523**

For all the values in **tables 1 and 2** (i.e. for all the values of m deriving from the numbers N≡ 3 mod 4) we have:

- ***Media ponderata*: 0.886293**
- ***Varianza ponderata*: 0,461207**
- ***Standard deviation* : 0,679122**

If we consider larger groups of integers between 0 and 2n-1 with n equal to 10 - 20 - 40 - 100 and repeat the calculation of the weighted mean value, variance and standard deviation, the following results are obtained in the following tables:

Point 5 of the sequence with p and k the same parity				
Value of p. Set of N numbers from 0 to 2^p-1	**Range of p**	**Weighted average m=2/(p−1)**	**Variance m=2/(p−1)**	**Standard deviation m=2/(p−1)**
10	2<=p<=10	1.38865	0.40511	0.63649
20	2<=p<=20	1.38630	0.40715	0.63808
40	2<=p<=40	1.38629	0.40715	0.63808
100	2<=p<=100	1.38629	0.40715	0.63808

Point 6 of the sequence with p and k different parity				
Value of p. Set of N numbers from 0 to 2^p-1	**Range of p**	**Weighted average m=1/p**	**Variance m=1/p**	**Standard deviation m=1/p**
10	2<=p<=10	0.38689	0.01511	0.12292
20	2<=p<=20	0.38630	0.01526	0.12352
40	2<=p<=40	0.38629	0.01526	0.12352
100	2<=p<=100	0.38629	0.01526	0.12352

Points 5 and 6 of the sequence, i.e. all numbers N≡ 3 mod. 4				
Value of p. Set of N numbers from 0 to 2^p-1	**Range of p**	**Weighted average m=2/(p-1) and m=1/p**	**Variance m=2/(p-1) and m=1/p**	**Standard deviation m=2/(p-1) and m=1/p**
10	2<=p<=10	0.88631	0.46117	0.67909
20	2<=p<=20	0.88629	0.46120	0.67912
40	2<=p<=40	0.88629	0.46120	0.67912
100	2<=p<=100	0.88629	0.46120	0.67912

Why don't the results change?

The stability of the results is due to the **weights** because each value of m has a weight proportional to 2^{n-p}, which decreases exponentially with the increase of p. This means:

- Values with small p (close to 2) have **dominant weight** where the first terms determine almost entirely the result.
- Values with p>20 and even more so with p>40, have **negligible weight**.
- Adding new terms with p up to 100 or more, results do not change significantly if we limit ourselves to values rounded to the first decimal places.

- The weighted average of about 0.886.
- The variance is relatively high (0.461), because the values range from 2.0000 to 0.0500, so the dispersion is large.
- The standard deviation of about 0.679 indicates that, on average, the values deviate by almost 0.7 from the weighted average but the probability of having numbers with very high p is very low very close to zero percent.

This distribution is stable for every p >=2 and therefore we can assert that **the results obtained remain valid for all odd positive integers with N ≡ 3 mod.4**.

Comparison between the weighted average of the angular coefficients m of line 1 and the angular coefficient m1=0.5849625 of line 2.

The mean value of the angular coefficient m=0.88629 of the segment of line 1 is greater than the angular coefficient m1=0.5849625 of the function f(ai) that defines the graph of line 1 because the weighted average is about 0.30 units higher than m1 and is a significant margin with respect to the scale of m-values. Furthermore, the **variance** of **0.461207** and the **standard deviation** of **0.679122** indicate that the values are **widely distributed** around the mean but nevertheless, since the mean is **much higher than m1**, even considering the dispersion, **most of the weight of the distribution is concentrated on values greater than m1.**
Furthermore, we remind you that in the personalized sequence in points 3 and 4 there are also the odd numbers N ≡ 1 mod.4 which have the segments of the line 1 corresponding to them vertical (point 3 with m = +∞) or with an angular coefficient equal to 1 (point 4 with m = 1) greater than m1 and therefore significantly increase the average value previously calculated for all the odd integers of the sequence.
In conclusion, segment 1 is formed by segments that have an average value significantly greater than m1=0.5849625 and this means **that, on average, the values of m obtained during the personalized sequence are on average higher than m1** and the predominance of segments with a slope greater than **m1** confirms that segment 1, as a whole, tends to grow faster than broken 2.
In the light of these results, the intersection theorem is strengthened: segment 1, composed of segments with an average slope greater than that of segment 2, inevitably crosses it at least one point. This intersection, formally demonstrated in paragraph 5, represents the moment when $b_s - a_s = f(a_s)$, that is, the convergence of the initial number to the value 1.N

7. *Formal proof of the intersection between the two broken lines*

Preliminary definitions

Let N be a positive integer. Consider a sequence (a_i, b_i) generated by an iterative process (customised Collatz sequence), equivalent to the Collatz sequence, where:

- a_i is the total number of odd steps taken up to and including iteration i;
- b_i is the total number of even steps taken up to and including iteration i;
- The difference $b_i - a_i$ always increases by 1 or 2 units with each iteration.

We define two broken lines in the Cartesian plane:

- **Broken line 1 obtained from the customised sequence**: (a_i, $b_i - a_i$) with starting point (0, 0)
- **Broken 2**: (a_i, $f(a_i)$) with starting point (0, $\log_2(N)$), defined by: $f(a_i) = \lceil a_{(i)} \cdot \lambda + \log_2(N) \rceil$ where

$\lambda = \log_2(3) \approx 0.5849625$ and $f(a_i)$ is rounded to the nearest positive integer.

Theorem of intersection between two broken lines

For every positive integer N, there exists at least one iteration s such that: $b_s - a_s = f(a_s)$ that is, the two broken lines intersect at the point: $(a_s, f(a_s)) = (a_s, b_s - a_s)$

Proof

1 **Growth of functions**

- The values of a_i do not decrease in successive iterations.
- The function $f(a_i)$ is strictly increasing with respect to a_i, being the sum of a strictly increasing linear function with slope $\lambda = \log_2(3) \approx 0.5849625$ and a constant $\log_2(N)$.
- The difference $b_i - a_i$ always increases by 1 or 2 units with each iteration.

2 **Discrete behaviour**

- Both broken lines are defined on discrete sets.
- Since $f(a_i)$ increases linearly and $b_i - a_i$ increases more rapidly (**on average**), there is at least one point where the two functions coincide.

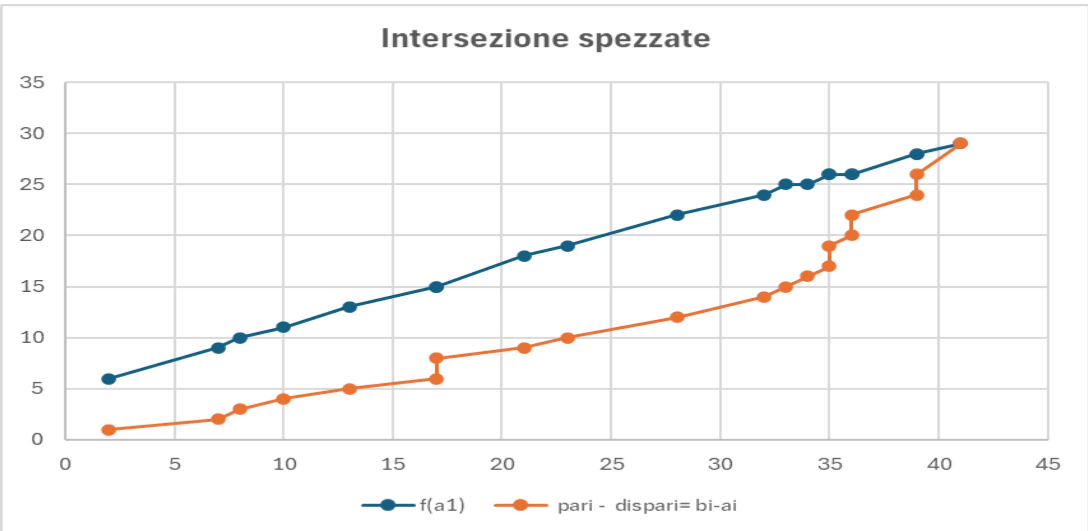
3 **Principle of discrete comparison**

- Initially: $a_0 = 0$, $b_0 = 0$, therefore $b_0 - a_0 = 0$
- But $f(a_0)$ = [$\log_2(N)$] > 0, therefore $b_0 - a_0$ < $f(a_0)$
- Then, due to the monotonicity of the two functions, there exists a first index s such that:

$$b_s - a_s = f(a_s) \quad \text{Q.E.D.}$$

Concrete example: *N* = 27

Collatz sequence using only odd numbers N≡1 mod 4 and N≡3 mod 4										
Iterations	step = Odd + Even	N	periodicity p or even and odd numbers	Value of 1+2*k	odd or even k	Transformation of N based on points 1 to 6	Odd a_i	Even b_i	Difference $a_i - b_i$	Rounded value of $f(a_i)$
		27								
1	5	31	2	7	3	Point 6	2	3	1	6
2	16	121	5	1	0	Point 6	7	9	2	9
3	19	91	1d2p			Point 4	8	11	3	9
4	24	103	2	23	11	Point 6	10	14	4	11
5	31	175	3	13	6	Point 6	13	18	5	12
6	40	445	4	11	5	Point 6	17	23	6	15
7	42	111	0d2p			Point 3	17	25	8	15
8	51	283	4	7	3	Point 6	21	30	9	17
9	56	319	2	71	35	Point 6	23	33	10	18
10	68	607	6	5	2	Point 5	28	40	12	21
11	78	769	5	19	9	Point 5	32	46	14	24
12	81	577	1d2p			Point 4	33	48	15	24
13	84	433	1d2p			Point 4	34	50	16	25
14	87	325	1d2p			Point 4	35	52	17	25
15	89	81	0d2p			Point 3	35	54	19	25
16	92	61	1d2p			Point 4	36	56	20	26
17	94	15	0d2p			Point 3	36	58	22	26
18	102	13	4	1	0	Point 5	39	63	24	28
19	104	3	0d2p			Point 3	39	65	26	28
20	111	1	2d5p	1	0	Exceptional case N=3	41	70	29	29



From the table of the customised sequence of N = 27 and the previous graph, we observe that:

- In each iteration ***i*** prior to ***s*** (***i*** < ***s***), the following inequality holds: $\boldsymbol{b_i - a_i < f(a_i)}$
  - at iteration ***s*** = 20: we have $a_s = 41$ $\quad b_s = 70$ $\quad b_s - a_s = 29$

  $f(a_s) = \lceil 41 \cdot 0.5849625 + \log_2(27) \rceil = \lceil 23.9834625 + 4.7548875 \rceil = \lceil 28.73835... \rceil = 29 \Rightarrow \boldsymbol{b_s - a_s = f(a_s)} \Rightarrow$ **intersection at point (41;29)**

The point of intersection between the two broken lines represents a balance between the empirical growth of the sequence and the theoretical estimate provided by the function $f(a_i)$. This point can be interpreted as a threshold where iterative complexity coincides with logarithmic prediction.

8. Interpretation of results

The application of the customised Collatz sequence compared to the canonical one reveals a significant difference in the number of iterations. This reduction is the result of optimised trajectory management, based on:

- **Module 4 classification**, which allows the type of orbit to be applied to be identified immediately;
- **Elimination of intermediate even steps**, which in the canonical sequence slow down the path towards unity;
- **Use of the concept of isopath**, which allows groups of numbers to be replaced with simpler equivalents without losing structural information.

Furthermore, the growing difference between the total number of even and odd steps demonstrates a structural trend that favours convergence: the constant increase of 1 or 2 units of $b_i - a_i$ for each iteration *i* not only speeds up the process, but suggests that the acceleration is systemic and not random. This certainly leads to the values $a_{(s)}$ *and* $b_{(s)} - a_{(s)}$ for which the two broken lines meet, and therefore the initial number N converges to 1 in iteration s.

9. Conclusions

If the customised sequence maintains this efficiency for arbitrary values of *N*, new scenarios open up for a constructive proof of Collatz's conjecture, based on combinatorial and modular criteria. In particular, the approach highlights how the information contained in numbers congruent to 1 and 3 (mod 4) is sufficient to describe the entire dynamics of the conjecture.

The customised Collatz sequence is not only an operational simplification, but also a potential theoretical key to addressing one of the most famous open problems in elementary mathematics.

This way of proceeding in the customised sequence leads to the same results as the canonical Collatz sequence and does not depend on the initial number, as it applies in the same way and leads to the same results on both integers with few digits and integers with an unlimited number of digits.

Many tests have been carried out on different numbers, including those with numerous digits, using a Python programme I created that faithfully follows my customised sequence. Tests carried out on numbers with a large number of digits have confirmed the validity of the conjecture.

10. References

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